J Membrane Biol (2014) 247:561–570
DOI 10.1007/s00232-014-9672-x

An Efficient Wavelet-Based Approximation Method to Gene Propagation Model Arising in Population Biology

R. Rajaraman · G. Hariharan

Received: 21 February 2014 / Accepted: 5 May 2014 / Published online: 8 June 2014

Abstract In this paper,we have applied an efficient wavelet-based approximation method for solving the Fisher's type and the fractional Fisher's type equations arising in biological sciences. To the best of our knowledge, until now there is no rigorous wavelet solution has been addressed for the Fisher's and fractional Fisher's equations. The highest derivative in the differential equation is expanded into Legendre series; this approximation is integrated while the boundary conditions are applied using integration constants. With the help of Legendre wavelets operational matrices, the Fisher's equation and the fractional Fisher's equation are converted into a system of algebraic equations. Block-pulse functions are used to investigate the Legendre wavelets coefficient vectors of nonlinear terms. The convergence of the proposed methods is proved. Finally, we have given some numerical examples to demonstrate the validity and applicability of the method.

Keywords Fisher's equation · Fractional Fisher's equation · Operational matrices · Legendre wavelets · Homotopy analysis method · Differential transform method

Introduction

Wavelet Analysis, as a relatively new area in applied mathematical research, has received considerable attention in dealing with PDEs and fractional type PDEs (Hariharan et al. 2009; Hariharan and Kannan 2009, 2010). The propagation of a mutant gene model was first introduced by Fisher, which is known as Fisher's equation (Murray 1977). These equations have wide applications in the fields of logistic population growth, flame propagation, euro physiology, autocatalytic chemical reactions, branching Brownian motion processes, and nuclear reactor theory (Hariharan and Kannan 2009; Wazwaz and Gorguis 2004; Olmos and Shizgal 2006). The Fisher–Kolmogorov equation describes the growth of a gene within a population. We have seen that the solution can easily be described as a traveling wave—moving with constant speed and without change of the front's shape. This means that the growth of the gene is the same at very time. We have used the leading edge approximation to the asymptotic behavior of left- and right- moving fronts. With an asymmetric derivative, we obtain different properties for both directions of propagation. The right-moving front is accelerated and again, the leading edge approximation has permitted to calculate the exponential speed. As for the Fisher–Kolmogorov model, we observe that for decreasing values of α, profiles accelerate later and at the beginning of the simulation, profiles seem to move with constant speed.

In recent years, wavelet transforms have found their way into many different fields in science and engineering. Moreover, wavelets established a connection with fast numerical algorithms.

Wavelet theory possesses many useful properties, such as Compact support, orthogonality, dyadic, orthonormality, and multi-resolution analysis (MRA). Fractional Partial differential equations (FPDEs) are generalizations of classical partial differential equations of integer order. Mathematical modeling of complex processes is a major challenge for the contemporary scientist. In contrast to simple classical systems, where the theory of integer order

R. Rajaraman · G. Hariharan (✉)
Department of Mathematics, School of Humanities & Sciences, SASTRA University, Thanjavur 613 401, Tamilnadu, India
e-mail: hariharang2011@gmail.com; hariharan@maths.sastra.edu

R. Rajaraman
e-mail: rraja@maths.sastra.edu

differential equations is sufficient to describe their dynamics, fractional derivatives provide an excellent and an efficient instrument for the description of memory and hereditary properties of various complex materials and systems (Turut and Guzel 2012; Meral et al. 2010; Seki et al. 2003; Henry and Wearne 2000; Baeumer et al. 2008; Rida et al. 2010; Momani and Qaralleh 2007). But these FPDEs are difficult to get their exact solutions (Turut and Guzel 2012; Meral et al. 2010; Seki et al. 2003; Cuyt and Wuytack 1987). So the approximation methods must be used. Analytical methods enable researchers to study the effect of different variables or parameters on the function under study easily. Recently, there are several new approaches have been proposed for solving nonlinear PDEs, for example, the Adomian decomposition method (Wazwaz and Gorguis 2004; Abdusalam 2004), the variational iteration method (Matinfar and Ghanbari 2009), differential Transform method (Matinfar et al. 2012), reduced differential transform method (Yıldırım et al. 2012), homotopy Analysis method (Khan et al. 2012; Liao 2004; Hariharan 2013), and exp-function method (He and Wu 2006). Recently, local fractional calculus has been used to deal with problems for non-differentiable functions; see (Yang 2011a, b, 2012; Yang and Baleanu 2013) and the references therein. Local fractional Fourier series method is one of very efficient and powerful techniques for finding the solutions of the local fractional differential equations. It is also worth noting that the advantage of the local fractional differential equations displays the non-differential solutions, which show the fractal and local behaviors of moments.

In recent years, nonlinear reaction diffusion equations (NLRDE) have been used as a basis for a wide variety of models, for the special spread of gene in population (Momani and Qaralleh 2007) and for chemical wave propagation (Hariharan and Kannan 2009, 2010). Wazwaz and Gorguis (2004) developed the Adomian decomposition Method for the Fisher type equations. Carey and Shen (1995) implemented the least square Finite element method for Fisher's reaction diffusion equation. Al-khaled (2001) introduced the sinc-collocation method by the Pseudo-spectral method for the numerical solution of Fisher's equation. Mittal and Ram (2008) have presented the differential quadrature method for Fisher's equations. Merdan (2012) solved the time-fractional reaction–diffusion equations by the fractional variational iteration method. Khan et al. (2012) established the analytical solutions of the fractional reaction–diffusion equations by the homotopy analysis method. Kurulay and Bayram (2012) showed the numerical solutions of time-fractional reaction–diffusion equation by the differential transform method. Yang et al. (2013) addressed a transient heat conduction problem in a fractal semi-infinite bar solved by the Yang-Fourier transform.

In the numerical analysis, wavelet-based methods and hybrid methods become important tools because of the properties of localization. In wavelet-based methods, there are two important ways of improving the approximation of the solutions: Increasing the order of the wavelet family and the increasing the resolution level of the wavelet. There is a growing interest in using various wavelets (Razzaghi and Yousefi 2000; Yousefi 2006; Mohammadi and Hosseini 2011; Maleknejad and Sohrabi 2007; Hariharan et al. 2009; Hariharan and Kannan 2009; Hariharan and Kannan 2010a, b; Yang 2013; Heydari et al. 2012) to study problems, of greater computational complexity. Among the wavelet transform families the Haar and Legendre wavelets deserve much attention. The basic idea of Legendre wavelet method is to convert the PDEs to a system of algebraic equations by the operational matrices of integral or derivative (Razzaghi and Yousefi 2001; Parsian 2005). The main goal is to show how wavelets and multi-resolution analysis can be applied for improving the method in terms of easy implementability and achieving the rapidity of its convergence. Razzaghi and Yousefi (Razzaghi and Yousefi 2001) introduced the Legendre wavelet method for solving variational problems and constrained optimal control problems. Hariharan et al. 2009, Hariharan and Kannan 2009, Hariharan and Kannan 2010a, b had introduced the diffusion equation, convection–diffusion equation, Reaction–diffusion equation, Non linear parabolic equations, fractional Klein–Gordon equations, Sine–Gordon equations and Fisher's equation by the Haar wavelet method. Mohammadi and Hosseini (2011) had showed a new Legendre wavelet operational matrix of derivative in solving singular ordinary differential equations. Jafari et al. (2011) had solved the fractional differential equations by the Lagendre wavelet method. Parsian (2005) introduced two dimension Legendre wavelets and operational matrices of integration. In recent years, many analytical/approximation methods have been proposed for solving Fisher's and fractional Fisher's equations. For example, Adomian decomposition method (Wazwaz and Gorguis 2004), the variational iteration method (Matinfar and Ghanbari 2009), the Homotopy perturbation method (Matinfar and Ghanbari 2009), the differential transform method (Matinfar et al. 2012), the homotopy analysis method (Hariharan 2013) and other methods (Olmos and Shizgal 2006). Recently, Hariharan and Rajaraman (2013) established a new coupled wavelet-based method applied to the nonlinear reaction–diffusion equation arising in mathematical chemistry. Yin et al. (2013) introduced a wavelet-based hybrid method for solving Klein–Gordan equations.

In this work, we have applied a wavelet-based coupled method (LLWM) which combines the Laplace transform method and the Legendre wavelets method for the numerical solution of Fisher's and fractional Fisher's equations.

This paper is organized as follows: Basic definitions of wavelets, Legendre wavelets and their properties are described in "Legendre wavelets and properties" section. Then, the method of solution of the Fisher's and fractional Fisher's equations by the LLWM is presented in "Methods of solution" section. In "Convergence analysis and error estimation" section, the convergence analysis is described. In "Illustrative examples" section, several numerical examples are presented to demonstrate the effectiveness of the proposed method. Concluding remarks are given in "Conclusion" section.

Legendre Wavelets and Properties

Wavelets

Wavelets are the family of functions which are derived from the family of scaling function $\{\phi_{j,k:} k \in Z\}$ where

$$\phi(x) = \sum_k a_k \phi(2x - k) \tag{1}$$

For the continuous wavelets, the following equation can be represented:

$$\Psi_{a,b}(x) = |a|^{\frac{-1}{2}} \Psi\left(\frac{x-b}{a}\right) \quad a,b \in R,\ a \neq 0 \tag{2}$$

where a and b are dilation and translation parameters, respectively, such that $\Psi(x)$ is a single wavelet function (Yin et al. 2012).

The discrete values are put for a and b in the initial form of the continuous wavelets, i.e.:

$$a = a_0^{-j},\ a_0 > 1,\ b_0 > 1 \tag{3}$$

$$b = kb_0 a_0^{-j}, \quad j,k \in Z. \tag{4}$$

Then, a family of discrete wavelets can be constructed as follows:

$$\Psi_{j,k} = |a_0|^{\frac{1}{2}} \Psi\left(2^j x - k\right) \tag{5}$$

So, $\Psi_{j,k}(x)$ constitutes an orthonormal basis in L^2 (R), where $\Psi(x)$ is a single function.

Legendre Wavelets

The Legendre wavelets are defined by

$$\psi_{nm}(t) = \begin{cases} \sqrt{m+\frac{1}{2}}\, 2^{\frac{k}{2}} L_m(2^k t - \widehat{n}), & \text{for } \frac{\widehat{n}-1}{2^k} \le t \le \frac{\widehat{n}+1}{2^k} \\ 0, & \text{otherwise} \end{cases}, \tag{6}$$

where $m = 0, 1, 2,\ldots, \mathrm{M} - 1$ and $k = 1, 2,\ldots, 2^{j-1}$. The coefficient $\sqrt{m+\frac{1}{2}}$ for orthonormality, then, the wavelets $\Psi_{k,m}(x)$ form an orthonormal basis for $L^2[0,1]$. In the above formulation of Legendre wavelets, the Legendre polynomials are in the following way:

$$\begin{aligned} p_0 &= 1, \\ p_1 &= x, \\ p_{m+1}(x) &= \frac{2m+1}{m+1} x p_m(x) - \frac{m}{m+1} p_{m-1}(x) \end{aligned} \tag{7}$$

and $\{p_{m+1}(x)\}$ are the orthogonal functions of order m, which is named the well-known shifted Legendre polynomials on the interval [0,1]. Note that, in the general form of Legendre wavelets, the dilation parameter is $a = 2^{-j}$ and the translation parameter is $b = n2^j$ (Yin et al. 2012).

Block-Pulse Functions (BPFs) (Yin et al. 2013)

The block-pulse functions form a complete set of orthogonal functions which defined on the interval [0, b) by

$$b_i(t) = \begin{cases} 1, & \frac{i-1}{m} b \le t < \frac{i}{m} b, \\ 0, & \text{elsewhere} \end{cases} \tag{8}$$

for $i = 1, 2,\ldots, m$. It is also known that for any absolutely integrable function $f(t)$ on [0, b) can be expanded in block-pulse functions:

$$f(t) \cong \xi^T B_m(t) \tag{9}$$

$$\xi^T = [f_1, f_2, \ldots, f_m],\ B_m(t) = [b_1(t), b_2(t), \ldots, b_m(t)] \tag{10}$$

where f_i are the coefficients of the block-pulse function, given by

$$f_i = \frac{m}{b} \int_0^b f(t) b_i(t) dt \tag{11}$$

Remark 1 Let A and B are two matrices of $m \times m$, then $A \otimes B = \left(a_{ij} \times b_{ij}\right)_{mm}$.

Lemma 1 *Assuming f(t) and g(t) are two absolutely integrable functions, which can be expanded in block-pulse function as f(t) = FB(t) and g(t) = GB(t), respectively, then we have*

$$f(t)g(t) = FB(t)B^T(t)G^T = HB(t) \tag{12}$$

where $H = F \otimes G$.

Approximating the Nonlinear Term (Yin et al. 2013)

The Legendre wavelets can be expanded into m-set of block-pulse Functions as

$$\Psi(t) = \phi_{m\times m} B_m(t) \tag{13}$$

Taking the collocation points as following

$$t_i = \frac{i - 1/2}{2^{k-1}M}, \quad i = 1, 2, \ldots, 2^{k-1}M \tag{14}$$

The m-square Legendre matrix $\phi_{m\times m}$ defined as

$$\phi_{m\times m} \cong [\Psi(t_1)\Psi(t_2)\ldots\Psi(t_{2^{k-1}M})] \tag{15}$$

The operational matrix of product of Legendre wavelets can be obtained using the properties of BPFs, let $f(x,t)$ and $g(x,t)$ are two absolutely integrable functions, which can be expanded by Legendre wavelets as $f(x,t) = \Psi^T(x)F\Psi(t)$ and $g(x,t) = \Psi^T(x)G\Psi(t)$, respectively.

$$f(x,t) = \Psi^T(x)F\Psi(t) = B^T(x)\phi^T_{mm}F\phi_{mm}B(t), \tag{16}$$

$$g(x,t) = \Psi^T(x)G\Psi(t) = B^T(x)\phi^T_{mm}G\phi_{mm}B(t), \tag{17}$$

and $F_b = \phi^T_{mm}F\phi_{mm}, G_b = \phi^T_{mm}G\phi_{mm}, H_b = F_b \otimes G_b$. Then,

$$\begin{aligned} f(x,t)g(x,t) &= B^T H_b B(t), \\ &= B^T(x)\phi^T_{mm} inv(\phi^T_{mm}) H_b inv(inv(\phi^T_{mm}) \\ H_b inv(\phi_{mm}))\phi_{mm}B(t) &= \Psi^T(x)H\Psi(t) \end{aligned} \tag{18}$$

where $H = inv(\phi^T_{mm})H_b inv((\phi_{mm}))$.

Function Approximation

A given function (x) with the domain [0, 1] can be approximated by

$$f(x) = \sum_{k=1}^{\infty}\sum_{m=0}^{\infty} c_{k,m}\Psi_{k,m}(x) = C^T.\Psi(x). \tag{19}$$

Here, C and Ψ are the matrices of size $(2^{j-1}\ M \times 1)$.

$$C = \left[c_{1,0}, c_{1,1}, \ldots c_{1,M-1}, c_{2,0}, c_{2,1}, \ldots c_{2,M-1}, \ldots c^{j-1}_{2,1}, \ldots, c^{j-1}_{2,M-1}\right]^T \tag{20}$$

$$\Psi(x) = [\Psi_{1,0}, \Psi_{1,1}, \Psi_{2,0}, \Psi_{2,1}, \ldots \Psi_{2,M-1}, \ldots \Psi_{2^{j-1},M-1}]^T \tag{21}$$

Method of Solution

Solving the Fisher's and Fractional Fisher's Equations by the LLWM

We consider the well-known Fisher's equation (Hariharan and Kannan 2009)

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} + \alpha U(1 - U) \tag{22}$$

with the initial condition

$$U(x,0) = f(x), \quad 0 \le x \le 1 \tag{23}$$

Taking Laplace transform on both sides of Eq. (22), we get

$$sL(U) - U(x,0) = L[U_{xx} + \alpha U - \alpha U^2] \tag{24}$$

$$sL(U) = U(x,0) + \left[L(U_{xx} + \alpha U - \alpha U^2)\right] \tag{25}$$

$$L(U) = \frac{U(x,0)}{s} + \frac{1}{s}L(U_{xx} + \alpha U - \alpha U^2) \tag{26}$$

Taking inverse Laplace transform to Eq. (26), we get

$$U(x,t) = U(x,0) + L^{-1}\left(\frac{1}{s}L\big(U_{xx} + \alpha U - \alpha U^2\big)\right) \tag{27}$$

Because

$$\begin{aligned} L^{-1}\left[\frac{1}{s}L(t^n)\right] &= L^{-1}\left(\frac{n!}{s^{n+2}}\right) \\ &= \frac{1}{n+1}t^{n+1}; \quad (n = 0, 1, 2, \ldots) \end{aligned} \tag{28}$$

We have

$$L^{-1}[s^{-1}L()] = \int_0^t (.)dt \tag{29}$$

From Eq. (27)

$$U(x,t) = U(x,0) + L^{-1}\left(\frac{1}{s}L(U_{xx} + g(U))\right) \tag{30}$$

where $g(U) = \alpha U - \alpha U^2$

$$U(x,t) = U(x,0) + L^{-1}\left(\frac{1}{s}L(U_{xx} + g(U))\right) \tag{31}$$

Using the Legendre wavelets method,

$$\left.\begin{aligned} U(x,t) &= C^T\psi(x,t) \\ U(x,0) &= S^T\psi(x,t) \\ g(U) &= G^T\psi(x,t) \end{aligned}\right\} \tag{32}$$

Substituting Eq. (32) in Eq. (27), we obtain

$$C^T = S^T + (C^T D_x^2 - G^T)P_t^2. \tag{33}$$

Here, G^T has a nonlinear relation with C. When we solve a nonlinear algebraic system, we get the solution is more complex and large computation time. In order to overcome the above drawbacks, we introduce an approximation formula as follows:

$$U_{n+1} = U(x,0) + \Pi\left[\frac{\partial^2 U_n}{\partial x^2} + g(U_n)\right] \tag{34}$$

where $g(U) = \alpha U - \alpha U^2$.

Expanding $u(x, t)$ by Legendre wavelets using the following relation

$$C_{n+1}^T = C_0^T + \left[C_n^T D_x^2 - G_n^T\right] P_t^2. \tag{35}$$

Convergence Analysis and Error Estimation (Yin et al. 2013; Hariharan and Rajaraman 2013)

$$U^* = U_0 + \Pi\left[U_{xx}^* + g(U^*)\right] \tag{36}$$

$$U_{n+1} = U_0 + \Pi\left[(U_n)_{xx} + g(U_n)\right] \tag{37}$$

Subtracting Eq. (36) from Eq. (37), we obtain

$$U_{n+1} - U^* = \Pi\left[(U_n - U^*)_{xx} + g(U_n) - g(U^*)\right] \tag{38}$$

Using Lipschitz condition, $\|g(U_n) - g(U^*)\| \le \gamma \|U_n - U^*\|$, we have

$$\|U_{n+1} - U^*\| \le \left\|\Pi(U_n - U^*)_{xx}\right\| + \|\Pi(g(U_n) - g(U^*))\| \tag{39}$$

$$\le \left\|\Pi(U_n - U^*)_{xx}\right\| + \gamma\|\Pi(U_n - U^*)\| \tag{40}$$

Let $U_{n+1} = C_{n+1}^T \psi(x, t)$

$U^* = C^T \psi(x, t)$

$\in_{n+1}^T = C_{n+!}^T - C^T$

Equation (40) gives

$$\in_{n+1}^T \le \in_n^T \left\|D_x^2 P_t^2 + \gamma P_t^2\right\| \tag{41}$$

The following formula Eq. (42) can be obtained using recursive relation.

$$\in_{n+1}^T \le \in_n^T \left\|D_x^2 P_t^2 + \gamma P_t^2\right\|^n \in_0 \tag{42}$$

When $\underset{n\to\infty}{Lim}\left\|D_x^2 P_t^2 + \gamma P_t^2\right\|^n = 0$, the series solution of Eq. (22) using the LLWM converges to $U^*(x)$. Using the definitions of D_x and P_t, we can get the value of γ.

Suppose $k = k' = 1$ and $M = M'$, the maximum element of D_x and P_t is $2\sqrt{(2M-1)(2M-3)}$ and 0.5, respectively.

Illustrative Examples

Example 1 We consider the Fisher's equation of the form (Hariharan and Kannan 2009)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha u(1 - u) \tag{43}$$

Subject to the initial condition

$$u(x, 0) = \frac{1}{\left(1 + e^{\sqrt{\frac{\alpha}{6}}x}\right)^2} \tag{44}$$

Using Homotopy analysis method (HAM) (see Appendix), the exact solution in a closed form is given by

$$u(x, t) = \frac{1}{\left(1 + e^{\sqrt{\frac{\alpha}{6}}x - \frac{5}{6}\alpha t}\right)^2} \tag{45}$$

The Haar wavelet scheme (HWS) of Eq. (43) is given by

$$c_{(m)}^T Q_{(m)} h_{(m)}(x_l) + x_l\left[-c_{(m)}^T P_{(m)}\lambda + g_1'(t_{s+1}) - g_0'(t_{s+1})\right] + g_0'(t_{s+1}) = u''(x_l, t_{s+1}) + \alpha u(x_l, t_{s+1})[1 - u(x_l, t_{s+1})]$$

Table 1 Comparison between the exact and LLWM for Example 1

x	t	U_{exact}	U_{LLWM}	Absolute error	Percentage (%) error
0.25	0.5	0.81839	0.81855	0.00016	0.012
	1.0	0.98292	0.98305	0.00013	0.013
	2.0	0.99988	0.99999	0.00011	0.011
	5.0	1.00000	1.00000	0.00000	0.000
0.50	0.5	0.77590	0.77602	0.00012	0.015
	1.0	0.97815	0.97824	0.00009	0.009
	2.0	0.99985	0.99996	0.00011	0.011
	5.0	1.00000	1.00000	0.00000	0.000
0.75	0.5	0.72582	0.72595	0.00013	0.017
	1.0	0.92207	0.92221	0.00014	0.015
	2.0	0.99981	0.99993	0.00012	0.012
	5.0	1.00000	1.00000	0.00000	0.000

Table 2 Comparison between the exact and LLWM for Example 2

x	t	U_{exact}	U_{LLWM}	Absolute error	Percentage (%) error
0.25	0.5	0.51830	0.51839	0.00009	0.017
	1.0	0.58011	0.58018	0.00007	0.012
	2.0	0.69492	0.69499	0.00007	0.001
	5.0	0.91078	0.91085	0.00007	0.007
	8.5	0.98331	0.98336	0.00005	0.005
	11.0	0.99513	0.99514	0.00001	0.001
0.50	0.5	0.47414	0.47423	0.00009	0.018
	1.0	0.53655	0.53661	0.00006	0.011
	2.0	0.65621	0.65626	0.00005	0.007
	5.0	0.89533	0.89535	0.00002	0.002
	8.5	0.98012	0.98015	0.00003	0.003
	11.0	0.99423	0.99424	0.00001	0.001
0.75	0.5	0.43037	0.43047	0.00010	0.023
	1.0	0.49242	0.49252	0.00010	0.020
	2.0	0.61531	0.61539	0.00008	0.013
	5.0	0.87757	0.87765	0.00008	0.009
	8.5	0.97633	0.97636	0.00003	0.003
	11.0	0.99312	0.99314	0.00002	0.002

Table 3 Comparison between the exact and LLWM for Example 3

x	t	U_{exact}	U_{LLWM}	Absolute error	Percentage (%) error
0.25	0.5	0.8184	0.8186	0.0002	0.02
	1.0	0.9829	0.9832	0.0003	0.03
	1.5	0.9999	0.9999	0.0000	0.00
	2.0	1.0000	1.0000	0.0000	0.00
0.50	0.5	0.7758	0.7761	0.0003	0.03
	1.0	0.9781	0.9783	0.0002	0.02
	1.5	0.9999	1.0000	0.0001	0.01
	2.0	1.0000	1.0000	0.0000	0.00
0.75	0.5	0.7258	0.7261	0.0003	0.04
	1.0	0.9721	0.9723	0.0002	0.02
	1.5	0.9998	0.9999	0.0001	0.01
	2.0	1.0000	1.0000	0.0000	0.00

Table 4 Comparison between exact solution and LLWM for Example 3 for different values of x and t

x	t	exact	LLWM	Absolute error	Percentage (%) error
0.1	0.2	0.5054	0.5062	0.0008	0.15
0.2	0.4	0.7364	0.7371	0.0007	0.09
0.3	0.6	0.8780	0.8786	0.0006	0.07
0.4	0.8	0.9475	0.9480	0.0005	0.05
0.5	1.0	0.9781	0.9784	0.0003	0.03
0.6	1.2	0.9910	0.9913	0.0003	0.03
0.7	1.4	0.9963	0.9966	0.0003	0.03
0.8	1.6	0.9985	0.9986	0.0001	0.01
0.9	1.8	0.9994	0.9994	0.0000	0.00
1.0	2.0	0.9998	0.9998	0.0000	0.00

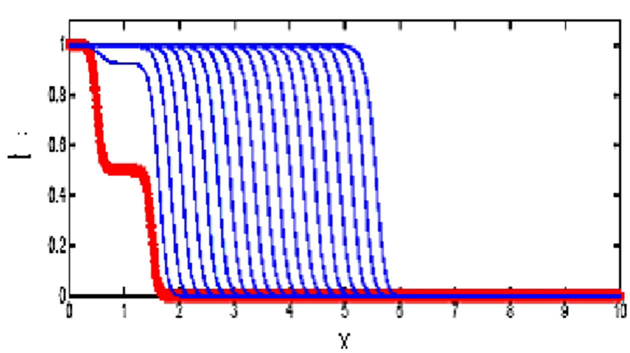


Fig. 1 Numerical solutions of Fisher's equation for (x, t) and $\alpha = 0.5$, $k = 1$ and $M = 4$

From the above formula, the wavelet coefficients $c^{T}_{(m)}$ can be successively calculated.

Our proposed method (LLWM) can be compared with Wazwaz and Gorguis results (See Ref. (Wazwaz and Gorguis 2004)), Mehmet Merdan results (See Ref. (Merdan 2012)), (Hariharan and Kannan 2009) and Zhou (Zhou 2008) results. Good agreement with the exact solution is observed.

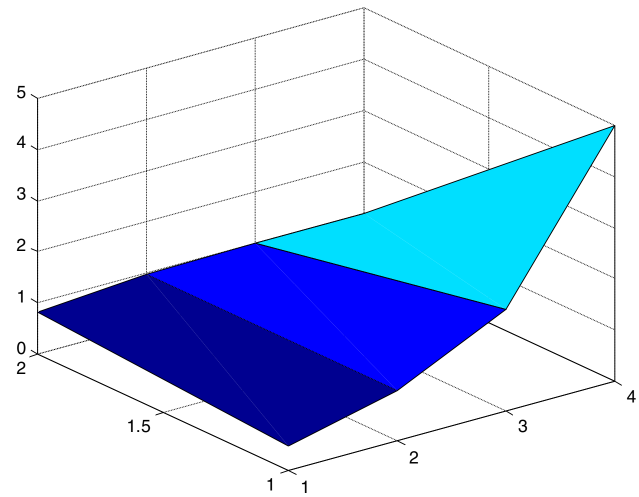


Fig. 2 The surface area shows that $u(x, t)$ using LLWM for Eq. (43) at $x = 0.25$, $k = 1$ and $M = 4$

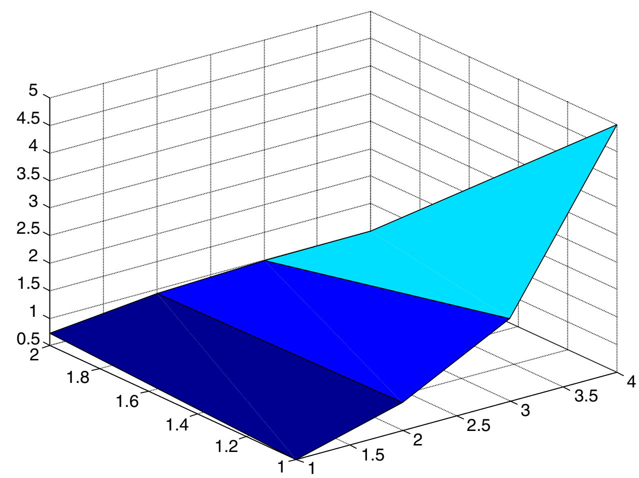


Fig. 3 The surface area shows that $u(x, t)$ using LLWM for Eq. (43) at $x = 0.75$, $k = 1$ and $M = 4$

Example 2 Consider the Fisher equation of the form (Wazwaz and Gorguis 2004; Yıldırım et al. 2012; Hariharan and Kannan 2009)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u^2(1-u), \quad 0 < x < 1 \tag{46}$$

With the initial condition

$$u(x,0) = \frac{1}{1 + e^{\frac{x}{\sqrt{2}}}} \tag{47}$$

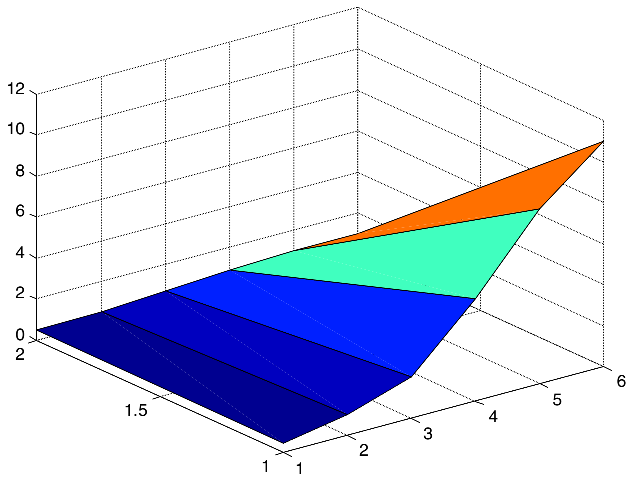


Fig. 4 The surface area shows that $u(x, t)$ using LLWM for Eq. (44) at $x = 0.25$, $k = 1$ and $M = 4$

Using the HAM, the exact solution in a closed form is given by

$$u(x,t) = \frac{1}{1+e^{v(x-vt)}}, \quad v = \frac{1}{\sqrt{2}} \tag{48}$$

Our proposed method (LLWM) can be compared with Wazwaz and Gorguis results (SeeRef. [26]) and Mehmet Merdan results (See Ref. Mehmet Merdan 2012), Hariharan et al. (2009) and Zhou (2008)results. Good agreement with the exact solution is observed.

Example 3 Let us consider the following time-fractional Fisher's reaction–diffusion equation

$$D_t^\alpha u = u_{xx} + 6u(1-u), \quad t > 0, \; x \in R \tag{49}$$

With initial condition

$$u(x,0) = \frac{1}{(1+e^x)^2} \tag{50}$$

Using differential transform method (DTM), the series solution is given by

$$\begin{aligned} u(x,t) = {} & \frac{1}{4} - \frac{1}{4}x + \frac{1}{16}x^2 + \frac{1}{48}x^3 \\ & + \left(\frac{5}{4} - \frac{5}{8}x - \frac{5}{16}x^2\right)\frac{t^\alpha}{\Gamma(\alpha+1)} \\ & + \left(\frac{25}{16} + \frac{25}{16}x\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{125}{8}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots \end{aligned} \tag{51}$$

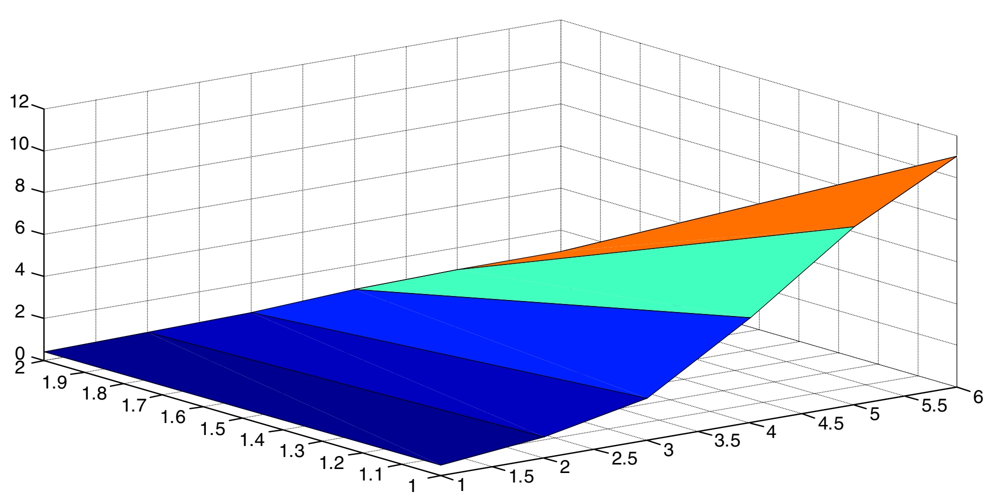
When $\alpha = 1$, the exact solution is given by

$$u(x,t) = \frac{1}{(1+e^{x-5t})^2} \tag{52}$$

Tables 1, 2, 3, and 4 show the numerical solutions of the Fisher's equations and the fractional Fisher's equations for various values (x, t) and $\alpha = 1$ Our LLWM results are in excellent agreement with the exact solution, the Homotopy analysis method (HAM) and the differential transform method (DTM). Figures 1, 2, 3, 4, 5 and 6 show the numerical solutions of the Fisher's equation and the fractional Fisher's equations for various values of (x, t) and $\alpha = 1$.

All the numerical experiments presented in this section were computed in double precision with some MATLAB codes on a personal computer System Vostro 1400 Processor ×86 Family 6 Model 15 Stepping 13 Genuine Intel ∼1596 MHz.

Fig. 5 The surface area shows that $u(x, t)$ using LLWM for Eq. (44) at $x = 0.75$, $k = 1$ and $M = 4$



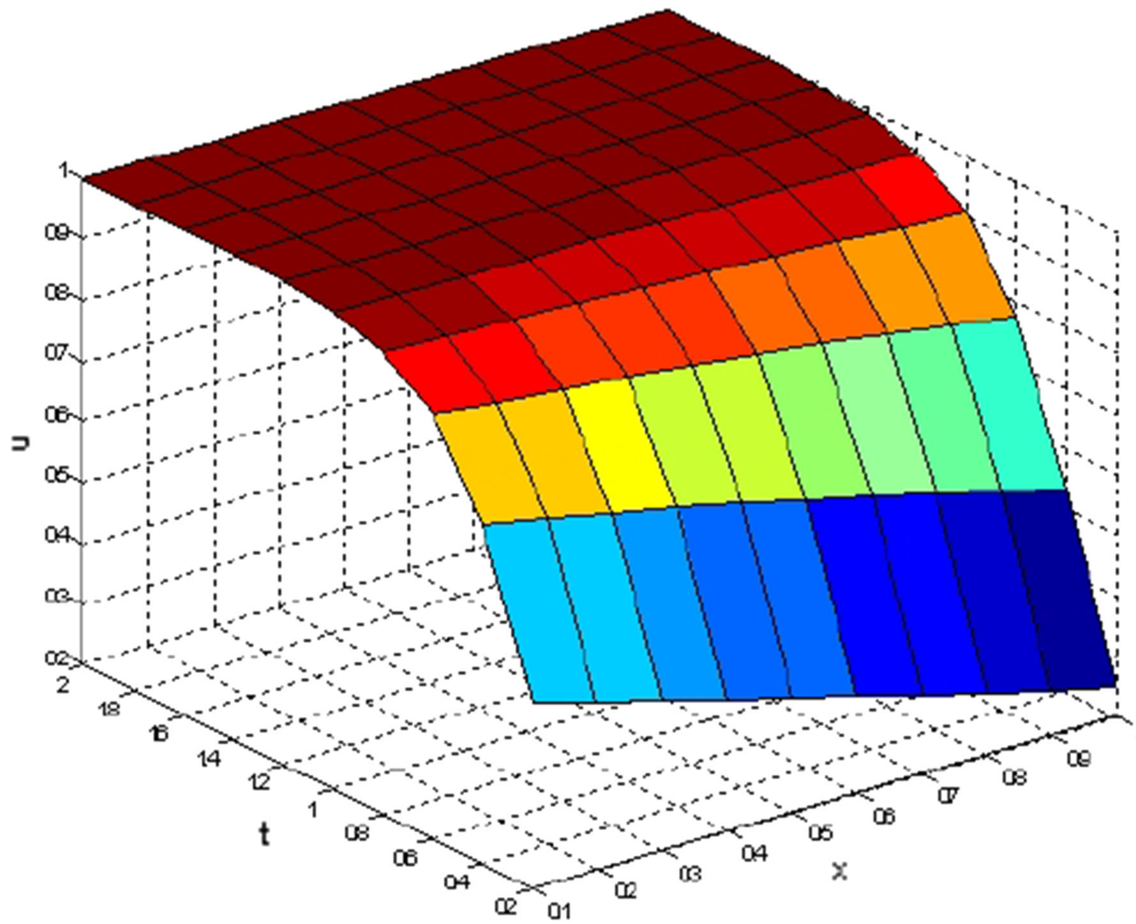


Fig. 6 The surface area shows that $u(x, t)$ using LLWM for Eq. (45) for various values of (x, t) and $M = 4$

Conclusion

In this work, a new coupled wavelet-based method has been successfully employed to obtain the numerical solutions of Fisher's and time-fractional Fisher's equations arising in population genetics. The proposed scheme is the capability to overcome the difficulty arising in calculating the integral values while dealing with nonlinear problems. This method shows higher efficiency than the traditional Legendre wavelet method for solving nonlinear PDEs. Numerical example illustrates the powerful of the proposed scheme LLWM. Also this paper illustrates the validity and excellent potential of the LLWM for nonlinear and fractional PDEs. The numerical solutions obtained using the proposed method show that the solutions are in very good coincidence with the exact solution. In addition, the calculations involved in LLWM are simple, straight forward and low computation cost. In section Convergence analysis and error estimation, we have developed the convergence of the proposed algorithm.

Acknowledgments This work was supported by the Naval Research Board (DNRD/05/4003/NRB/322), Government of India. The author is very grateful to the referees for their valuable suggestions. Our hearty thanks are due to Prof. R. Sethuraman, Vice-Chancellor, SASTRA University, Dr. S. Vaidhyasubramaniam, Dean/Planning and development and Dr. S. Swaminathan, Dean/Sponsored research for their kind encouragement and for providing good research environment.

Appendix

Basic Idea of Homotopy Analysis Method (HAM)

In this section, the basic ideas of the homotopy analysis method are presented. Here, a description of the method is given to handle the general nonlinear problem.

$$Nu_0(t) = 0, \quad t > 0 \tag{53}$$

where N is a nonlinear operator and $u_0(t)$ is unknown function of the independent variable t.

Zero-Order Deformation Equation

Let $u_0(t)$ denote the initial guess of the exact solution of Eq. (1), $h \neq 0$ an auxiliary parameter, $H(t) \neq 0$ an auxiliary function and L is an auxiliary linear operator with the property.

$$L(f(t)) = 0, \quad f(t) = 0 \tag{54}$$

The auxiliary parameter h, the auxiliary function $H(t)$, and the auxiliary linear operator L play an important role within the HAM to adjust and control the convergence region of solution series. Liao (Merdan 2012) constructs, using $q \in [0, 1]$ as an embedding parameter, the so-called zero-order deformation equation.

$$(1-q)L[(\phi(t;q) - u_0(t)] = qhH(t)N[(\phi(t;q)], \tag{55}$$

where $\phi(t;q)$ is the solution which depends on h, $H(t), L$, $u_0(t)$ and q. When $q = 0$, the zero-order deformation Eq. (54) becomes

$$\phi(t;0) = u_0(t), \tag{56}$$

and when $q = 1$, since $h \neq 0$ and $H(t) \neq 0$, the zero-order deformation Eq. (53) reduces to,

$$N[\phi(t;1)] = 0, \tag{57}$$

So, $\phi(t;1)$ is exactly the solution of the nonlinear equation. Define the so-called mth order deformation derivatives.

$$u_m(t) = \frac{1}{m!}\frac{\partial^m \phi(t;q)}{\partial q^m} \tag{58}$$

If the power series Eq. (55) of $\phi(t;q)$ converges at q = 1, then we gets the following series solution:

$$u(t) = u_0(t) + \sum_{m=1}^{\infty} u_m(t) \tag{59}$$

where the terms $u_m(t)$ can be determined by the so-called high-order deformation described below.

High-Order Deformation Equation

Define the vector,

$$\overrightarrow{u_{n}}{=}\{u_0(t), u_1(t), u_2(t)\ldots\ldots u_n(t)\} \tag{60}$$

Differentiating Eq. (55) m times with respect to embedding parameter q, the setting $q = 0$ and dividing them by !, we have the so-called mth order deformation equation.

$$L[u_m(t) - \aleph_m u_{m-1}(t)] = hH(t)R_m(\overrightarrow{u_m}, t), \tag{61}$$

where

$$\aleph_m = \begin{cases} 0, & m \leq 1 \\ 1, & otherwise \end{cases} \tag{62}$$

and

$$R_m(\overrightarrow{u_m}, t) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\phi(t;q)]}{\partial q^{m-1}} \tag{63}$$

For any given nonlinear operator N, the term $R_m(\overrightarrow{u_m}, t)$ can be easily expressed by Eq. (63). Thus, we can gain $u_1(t), u_2(t)\ldots\ldots$ by means of solving the linear high-order deformation with one after the other order in order. The mth order approximation of $u(t)$ is given by

$$u(t) = \sum\nolimits_{k=0}^{m} u_k(t) \tag{64}$$

ADM, VIM and HPM are special cases of HAM when we set $h = -1$ and $H(r, t) = 1$. We will get the same solutions for all the problems by above methods when we set $h = -1$ and $H(r, t) = 1$. When the base functions are introduced the $H(r, t) = 1$ is properly chosen using the rule of solution expression, rule of coefficient of ergodicity and rule of solution existence.

References

Abdusalam HA (2004) Analytic and approximate solutions for Nagumo telegraph reaction diffusion equation. Appl Math Comput 157:515–522

Al-Khaled K (2001) Numerical study of Fisher's reaction–diffusion equation by the Sinc collocation method. J Comput Appl Math 137:245–255

Baeumer B, Kovács M, Meerschaert MM (2008) Numerical solutions for fractional reaction–diffusion equations. Comput Math Appl 55(10):2212–2226

Carey GF, Shen Y (1995) Least-squares finite element approximation of Fisher's reaction–diffusion equation. Numer Methods Partial Differ Equ 11(2):175–186

Cuyt A, Wuytack L (1987) Nonlinear methods in numerical analysis. Elsevier Science, Amsterdam

Hariharan G (2013) The homotopy analysis method applied to the Kolmogorov–Petrovskii–Piskunov (KPP) and fractional KPP equations. J Math Chem 51:992–1000. doi:10.1007/s10910-012-0132-5

Hariharan G, Kannan K (2009) Haar wavelet method for solving Fisher's equation. Appl Math Comput 211:284–292

Hariharan G, Kannan K (2010a) Haar wavelet method for solving nonlinear parabolic equations. J Math Chem 48:1044–1061

Hariharan G, Kannan K (2010b) A comparative study of a Haar wavelet method and a restrictive taylor's series method for solving convection–diffusion equations. Int J Comput Methods Eng Sci Mech 11(4):173–184

Hariharan G, Rajaraman R (2013) A new coupled wavelet-based method applied to the nonlinear reaction–diffusion equation arising in mathematical chemistry. J Math Chem 51:2386–2400. doi:10.1007/s10910-013-0217-9

Hariharan G, Kannan K, Sharma K (2009) Haar wavelet in estimating the depth profile of soil temperature. Appl Math Comput 210:119–225

He JH, Wu XH (2006) Exp-function method for nonlinear wave equations. Chaos Solitons Fractals 30:700–708

Henry BI, Wearne SL (2000) Fractional reaction–diffusion. Physica A 276(3–4):448–455

Heydari MH, Hooshmandasl MR, Maalek Ghaini FM Mohammadi F (2012) Wavelet collocation method for solving multiorder fractional differential equations. J Appl Math Article ID 54240. doi:10.1155/2012/542401

Jafari H, Soleymanivaraki M, Firoozjaee MA (2011) Legendre wavelets for solving fractional differential equations. J Appl Math 7(4):65–70

Khan NA, Khan NU, Ara A, Jamil M (2012) Approximate analytical solutions of fractional reaction–diffusion equations. J King Saud Univ Sci 24:111–118

Kurulay M, Bayram M (2012) Comparison of numerical solutions of time-fractional reaction–diffusion equations. Malays J Math Sci 6(S):49–59

Liao SJ (2004) Beyond perturbation: introduction to homotopy analysis method. CRC Press/Chapman and Hall, Boca Raton

Maleknejad K, Sohrabi S (2007) Numerical solution of Fredholm integral equations of the first kind by using Legendre wavelets. Appl Math Comput 186:836–843

Matinfar M, Ghanbari M (2009a) Solving the Fishers equation by means of variational iteration method. Int J Contemp Math Sci 4(7):343–348

Matinfar M, Ghanbari M (2009b) Homotopy perturbation method for the Fisher's equation and its generalized. Int J Nonlinear Sci 8(4):448–455

Matinfar M, Bahar SR, Ghasemi M (2012) Solving the generalized Fisher's equation by the differential transform method. J Appl Math Inform 30(3–4):555–560

Meral FC, Royston TJ, Magin R (2010) Fractional calculus in viscoelasticity: an experimental study. Commun Nonlinear Sci Numer Simul 15(4):939–945

Merdan M (2012) Solutions of time-fractional reaction–diffusion equation with modified Riemann–Liouville derivative. Int J Phys Sci 7(15):2317–2326

Mittal RC, Ram J (2008) Numerical study of Fisher's equation by using differential quadrature method. Int J Inf Syst Sci 5(1):143–160

Mohammadi F, Hosseini MM (2011) A new Legendre wavelet operational matrix of derivative and its applications in solving the singular ordinary differential equations. J Franklin Inst 348:1787–1796

Momani S, Qaralleh R (2007) Numerical approximations and Pade approximants for a fractional population growth model. Appl Math Model 31(9):1907–1914

Murray JD (1977) Lectures on non-linear differential equation models in biology. Clarenden Press, Oxford

Olmos D, Shizgal B (2006) A spectral method of solution of Fisher's equation. J Comput Appl Math 193:219–242

Parsian H (2005) Two dimension Legendre wavelets and operational matrices of integration. Acta Math Acad Paedagog Nyiregyhazi 21:101–106

Razzaghi M, Yousefi S (2000) The Legendre wavelets direct method for variational problems. Math Comput Simul 53:185–192

Razzaghi M, Yousefi S (2001) The Legendre wavelets operational matrix of integration. Int J Syst Sci 32:495–502

Rida SZ, El-Sayed AMA, Arafa AAM (2010) On the solutions of time-fractional reaction–diffusion equations. Commun Nonlinear Sci Numer Simul 5(12):3847–3854

Seki K, Wojcik M, Tachiya M (2003) Fractional reaction–diffusion equation. J Chem Phys 119:2165–2174

Turut V, Guzel N (2012) Comparing numerical methods for solving time-fractional reaction–diffusion equations. ISRN Math Anal Article ID 737206. doi:10.5402/2012/737206

Wazwaz AM, Gorguis A (2004) An analytical study of Fisher's equation by using Adomian decomposition method. Appl Math Comput 154:609–620

Yang XJ (2011a) Local fractional functional analysis and its applications. Asian Academic, Hong Kong

Yang XY (2011b) Local fractional integral transforms. Prog Nonlinear Sci 4 (2011):1–225

Yang XJ (2012) Advanced local fractional calculus and its applications. World Science, New York

Yang Y (2013) Solving a nonlinear multi-order fractional differential equation using Legendre pseudo-spectral method. Appl Math 4:113–118. doi:10.4236/am.2013.41020

Yang XJ, Baleanu D (2013) Fractal heat conduction problem solved by local fractional variation iteration method. Therm Sci 17(2):625–628

Yang A-M, Zhang Y-Z, Long Y (2013) The Yang-Fourier transforms to heat-conduction in a semi-infinite fractal bar. Thermal Science 17(3):707–713

Yıldırım K, İbiş B, Bayram M (2012) New solutions of the nonlinear Fisher type equations by the reduced differential transform. Nonlinear Sci Lett A 3(1):29–36

Yin F, Song J, Lu F, Leng H (2012) A coupled method of laplace transform and Legendre wavelets for Lane–Emden-type differential equations. J Appl Math 2012, Article ID 163821. doi:10.1155/2012/163821

Yin F, Song J, Lu F (2013) A coupled method of Laplace transform and Legendre wavelets for nonlinear Klein–Gordon equations. Math Methods Appl Sci. doi: 10.1002/mma.2834

Yousefi SA (2006) Legendre wavelets method for solving differential equations of Lane–Emden type. App Math Comput 181:1417–1442

Zhou XW (2008) Exp-function method for solving Fisher's equation. J Phys Conf Ser 96:012063